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Noise Bias in Various Formulations of Ibrahim's Time Domain Technique

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Introduction

IBRAHIM's Time Domain Technique (ITD) estimates natural frequencies, damping factors, and mode shapes using sampled, free response data in the time domain. In the ITD, either the position, velocity, or acceleration is measured at n locations (for a system with n degrees of freedom) at discrete time intervals. These measurements are used to form the coefficient matrix of an eigenvalue problem. The eigenvalues and eigenvectors of this matrix can be directly related to the natural frequencies, damping factors, and mode shapes. If there is no noise in the measurements, then only $2n$ discrete time samples are needed to form a coefficient matrix to precisely determine the vibration parameters. More data is required when noise is present in order to estimate the coefficient matrix in a least squares (LS) sense.[1] However, it has been recognized that often the LS algorithms result in biased estimation, and therefore alternate methods such as Maximum Likelihood and Instrumental Variables have been proposed to eliminate the bias problems.[2] Reference 3 presents a double least squares (DLS) procedure in an effort to reduce the bias in the LS ITD technique. This study compares the results from the DLS and the LS techniques with an unbiased Instrumental Variable (IV) approach. The analytical expressions for the bias terms in the LS and the DLS ITD methods are provided.

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ITD Theory

The equations of motion for the free response of a discrete, multiple-degree-of-freedom, damped system can be written as

$$M\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2}+C\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t}+K\boldsymbol{x}=0 \tag{1}$$

where $\boldsymbol{x}$ is a vector of responses at n assumed degrees of freedom. The solutions of these equations are of the form $\boldsymbol{x}(t)=\boldsymbol{p}\exp(\lambda t)$. Substitution into Eq. (1) yields the eigenvalue problem

$$[\lambda^2 M+\lambda C+K]\boldsymbol{p}=0 \tag{2}$$

There exist $2n$ eigenvalues, λ. Here, $\boldsymbol{p}$ represents the complex modes of vibration. The system response is found by the summation, and the mode shape scaling depends upon the initial conditions

$$\boldsymbol{x}(t)=\sum_{j=1}^{2n}\boldsymbol{p}_j\exp(\lambda_j t) \tag{3}$$

Application of the ITD technique[1] for "noise-free" data results in the expression

$$\Phi'\Phi^{-1}\Psi_i=\exp(\lambda_i\Delta t)\Psi_i \tag{4}$$

where Φ and Φ' are developed from $2n$ discrete time values of $\boldsymbol{x}(t)$. The complex eigenvalues and eigenvectors of $\Phi'\Phi^{-1}$ can be used to describe completely the vibration parameters of the system.

Measurements with Noise

Experimental data usually is contaminated with noise, and therefore the eigenvalue problem must be modified to deal with noise. The modification proposed in Ref. 1 uses an LS algorithm and, though it was recognized that the LS estimates were biased,[3] the analytical expression for the bias was not provided. The bias led to a DLS approach[3] in an effort to reduce the influence of the noise on the parameter estimates.

Consider the case where the response data is corrupted with noise. The measured response is

$$\boldsymbol{x}_m(t)=\sum_{j=1}^{2n}\boldsymbol{p}_j\exp(\lambda_j t)+\boldsymbol{n}(t) \tag{5}$$

where $\boldsymbol{n}(t)$ is a stochastic vector comprised of the noise process at each measurement location. An overdetermined set of equations is needed for the LS algorithm. Measurements at r

time instances are used, where $r>2n$. Equation (5) is written r times in matrix form as

$$[x_m(t_1)x_m(t_2)...x_m(t_r)] = [p_1p_2\cdots p_{2n}]$$

$$[e(t_1)e(t_2)...e(t_r)] + [n(t_1)n(t_2)...n(t_r)] \tag{6}$$

or

$$X = P\Lambda + N_x \tag{7}$$

where

$$e(t) = [\exp(\lambda_1 t)\ \exp(\lambda_2 t)...\exp(\lambda_{2n} t)]^T \tag{8}$$

Defining $x_m(t+\Delta t) = y_m(t)$, then one can write

$$Y = PD\Lambda + N_y \tag{9}$$

where

$$D = \mathrm{diag}[\exp(\lambda_1\Delta t),\quad \exp(\lambda_2\Delta t),...,\exp(\lambda_{2n}\Delta t)] \tag{10}$$

Equations (8) and (9) can be combined to form

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} P \\ PD \end{bmatrix} \Lambda + \begin{bmatrix} N_x \\ N_y \end{bmatrix} \tag{11}$$

or

$$\Phi = \Psi\Lambda + N \tag{12}$$

Likewise, if we define $y_m(t+\Delta t) = z_m(t)$, then

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = \begin{bmatrix} PD \\ PD^2 \end{bmatrix} \Lambda + \begin{bmatrix} N_y \\ N_z \end{bmatrix} \tag{13}$$

or

$$\Phi' = \Psi'\Lambda + N' \tag{14}$$

In the LS approach, Eqs. (12) and (14) need to be solved and put into a form similar to Eq. (4). Since Ψ is invertable,[1] Eq. (12) can be solved for Λ and substituted into Eq. (14) to yield

$$\Phi' - \Psi'\Psi^{-1}\Phi = N' - \Psi'\Psi^{-1}N \tag{15}$$

Letting $A = \Psi'\Psi^{-1}$, then Eq. (15) can be rearranged as

$$\Phi' - A\Phi = E_a \tag{16}$$

where $E_a = N' - AN$. Minimization of $\|E_a\|$ with respect to A yields

$$A = \Phi'\Phi^t(\Phi\Phi^t)^{-1} \tag{17}$$

which, combined with the definition of A, is rearranged to yield Ibrahim's LS estimate

$$\Phi'\Phi^t(\Phi\Phi^t)^{-1}\Psi = \Psi' \tag{18}$$

Recognizing that $\Psi' = \Psi D$, the LS eigenvalue problem becomes

$$\Phi'\Phi^t(\Phi\Phi^t)^{-1}\Psi_i = \exp(\lambda_i\Delta t)\Psi_i \tag{19}$$

where Ψ_i represents the ith column of Ψ. The bias in the LS technique is apparent by postmultiplying Eq. (15) by $\Phi^t(\Phi\Phi^t)^{-1}\Psi$ and rearranging to yield

$$\{\Phi'\Phi^t(\Phi\Phi^t)^{-1} + \Psi D\Psi^{-1}N\Phi^t(\Phi\Phi^t)^{-1}$$

$$-N'\Phi^t(\Phi\Phi^t)^{-1}\}\Psi = \Psi' \tag{20}$$

The bias in estimating A by Eq. (17) is recognized in Eq. (20) and can be rewritten as

$$A_{\mathrm{bias}} = \Psi D\Psi^{-1}[(1/r)N\Phi^t][(1/r)\Phi\Phi^t]^{-1}$$

$$-[(1/r)N'\Phi^t][(1/r)\Phi\Phi^t]^{-1} \tag{21}$$

which does not approach zero in probability as r approaches infinity because

$$P\lim_{r\to\infty}\left[\frac{1}{r}N\Phi^t\right] \neq 0$$

$$P\lim_{r\to\infty}\left[\frac{1}{r}N'\Phi^t\right] \neq 0 \tag{22}$$

$$P\lim_{r\to\infty}\left[\frac{1}{r}\Phi\Phi^t\right] = G$$

An alternate LS solution exists. Ibrahim uses the alternative LS formulation along with the preceding LS solution in a DLS technique.[3] The alternate formulation solves Eqs. (12) and (14) in an LS sense by first solving Eq. (14) for Λ and then substituting into Eq. (12). After an LS minimization process, the alternate LS solution yields the eigenvalue problem

$$\Phi'\Phi'^t(\Phi\Phi'^t)^{-1}\Psi_i = \exp(\lambda_i\Delta t)\Psi_i \tag{23}$$

The bias in estimating A in the alternate solution can be shown to be

$$A_{\mathrm{bias}} = \Psi D\Psi^{-1}[(1/r)N\Phi'^t][(1/r)\Phi\Phi'^t]^{-1}$$

$$-[(1/r)N'\Phi'^t][(1/r)\Phi\Phi'^t]^{-1} \tag{24}$$

which also does not approach zero asymptotically by Eq. (22). Reference 3 suggests that the bias in the LS eigenvalue problem of Eq. (19) is opposite in sign to the bias in the alternate LS eigenvalue problem of Eq. (23) and suggests combining the two as a DLS procedure to produce

$$\tfrac{1}{2}\{\Phi'\Phi^t(\Phi\Phi^t)^{-1} + \Phi'\Phi'^t(\Phi\Phi'^t)^{-1}\}\Psi_i$$

$$= \exp(\lambda_i\Delta t)\Psi_i \tag{25}$$

Reference 3 demonstrates results using experimental data that support the assumption that the bias of the DLS problem is less than the bias of the first LS solution. The bias of the DLS solution is determined to be

$$A_{\mathrm{bias}} = \tfrac{1}{2}[\Psi D\Psi^{-1}N\{\Phi^t(\Phi\Phi^t)^{-1} + \Phi'^t(\Phi\Phi'^t)^{-1}\}$$

$$-N'\{\Phi^t(\Phi\Phi^t)^{-1} + \Phi'^t(\Phi\Phi'^t)^{-1}\}] \tag{26}$$

It is difficult to tell if Eq. (26) represents a reduction in the bias.

Instrumental Variable (IV) Approach

The LS procedures used the pseudo-inverses $\Phi^t(\Phi\Phi^t)^{-1}$ or $\Phi'^t(\Phi\Phi'^t)^{-1}$ to minimize certain quantities while solving an overdetermined set of equations. The bias was the result of nonzero expectations of certain terms in Eq. (21). However, if a matrix Ξ could be found so that the following were satisfied,

$$P\lim_{r\to\infty}\left[\frac{1}{r}N\Xi^t\right] = 0$$

$$P\lim_{r\to\infty}\left[\frac{1}{r}N'\Xi^t\right] = 0 \tag{27}$$

$$P\lim_{r\to\infty}\left[\frac{1}{r}\Phi\Xi^t\right] = G$$

Table 1 System parameters for numerical simulation

ω_d, rad/s	444.8	1983.1	4485.2	7980.5
Damping factor, ξ	0.453	0.1133	0.0504	0.0283
Mode shape				
y_1	1.0000	1.0000	1.0000	1.0000
y_2	2.4142	1.0000	−0.4142	−1.0000
y_3	2.4142	−1.0000	−0.4142	1.0000
y_4	1.0000	−1.0000	1.0000	−1.0000

Table 2 Estimated frequencies and damping factors (SNR = 60 dB)

LS	ω_d	444.2	1984	4486	7979
	ζ	0.465	0.1144	0.0510	0.0283
DLS	ω_d	443.7	1985	4483	7980
	ζ	0.457	0.1138	0.0511	0.0282
IV_{poor}	ω_d	444.7	1984	4491	7978
	ζ	0.458	0.1138	0.0502	0.0293
$IV_{perfect}$	ω_d	445.0	1984	4487	7980
	ζ	0.447	0.1137	0.0506	0.0284

Table 3 Estimated frequencies and dampings, SNR = 40 dB

LS	ω_d	233.4	1995	4495	7969
	ζ	0.904	0.1389	0.0589	0.0290
DLS	ω_d	457.0	2016	4468	7978
	ζ	0.327	0.1167	0.0555	0.0266
IV_{poor}	ω_d	443.6	1989	4544	7959
	ζ	0.505	0.1189	0.0490	0.0382
$IV_{perfect}$	ω_d	444.7	1993	4506	7972
	ζ	0.388	0.1174	0.0523	0.0293

Table 4 Estimated frequencies and dampings, SNR = 20 dB

LS	ω_d	572.7	1478	4109	7966
	ζ	0.989	0.6935	0.2994	0.0809
DLS	ω_d	—	2208	4184	7719
	ζ		−0.7557	−0.0896	−0.0498
IV_{poor}	ω_d	—	1783	4839	7550
	ζ		0.1889	0.0221	0.03662
$IV_{perfect}$	ω_d	348.8	2162	4715	7917
	ζ	−0.0407	0.131	0.0890	0.0322

where G is nonsingular, then a pseudo-inverse of $\Xi^t(\Phi\Xi^t)^{-1}$ would provide unbiased estimates.[2,4] This is shown by post-multiplying Eq. (15) by $\Xi^t(\Phi\Xi^t)^{-1}\Psi$, which yields

$$\{\Phi'\Xi^t(\Phi\Xi^t)^{-1}+\Psi'\Psi^{-1}N\Xi^t(\Phi\Xi^t)^{-1} -N'\Xi^t(\Phi\Xi^t)^{-1}\}\Psi=\Psi' \tag{28}$$

The bias asymptotically approaches zero by Eq. (27), and the IV estimate of the eigenvalue problem is

$$\Phi'\Xi^t(\Phi\Xi^t)^{-1}\Psi_i=\exp(\lambda_i\Delta t)\Psi_i \tag{29}$$

Here, Ξ is referred to as the Instrumental Variable Matrix. The difficulty associated with the use of the IV method is the generation of the instrumental variables.

ITD Numerical Experiments

Numerical simulations were conducted to compare the three methods discussed for solving the Ibrahim Time Domain method. The difference in the methods is the way the eigenvalue problem matrix is formed. Extra computational modes were used with the LS and DLS methods to reduce the effect of noise upon the ITD method.[5] The simulations were grouped as 1) the LS method (LS) with extra computational modes: Eq. (19), 2) the DLS method with extra computational modes: Eq. (25), and 3) the IV approach with no extra computational modes: Eq. (29).

Simulated free vibration, time domain response data was generated using the system parameters listed in Table 1. These parameters correspond to those used in Ref. 6. In choosing the simulation vibration parameters, one should take extra care to ensure that the mode shapes are indeed orthogonal. If the shapes are not orthogonal, then numerical difficulties will arise when inverting the matrices. The Nyquist frequency was selected to be 10,000 rad/s so that no aliasing of the frequencies would occur during identification. The simulated response was generated in a modal fashion, and the response vector was corrupted with a Gaussian noise vector at various levels. The oversized identifications used four additional simulated pseudomeasurement locations (see Refs. 1 and 5), also corrupted with simulated noise, to provide the extra computational modes. The signal-to-noise (SNR) used were 60, 40, and 20 dB, with SNR defined as

$$\text{SNR}=20\log(\text{abs}(A)/\sigma_n) \tag{30}$$

where σ_n is the standard deviation of the noise and A the amplitude of the response, which was approximately the peak amplitude of the response at the beginning of the time record. The three methods then were used to estimate the damped natural frequencies ω_d and damping factors ζ.

The IV method requires the generation of instrumental variables. These variables should be uncorrelated with the noise and correlated with the actual response in order to satisfy Eq. (27). The instrumental variables used in the simulation were generated using the exact mode shapes of response. The natural frequencies used in the instrumental variable generation were nine-tenths of the exact natural frequencies (labeled IV_{poor} in the Tables) or the exact natural frequencies (labeled $IV_{perfect}$ in the Tables). Obviously, when the exact frequencies are used, the instrumental variables are perfectly correlated with the exact response. Both sets of instrumental variables are uncorrelated with the added Gaussian noise.

Tables 2–4 show the simulation results for SNR's of 60, 40, and 20 dB, respectively. For high SNR, Table 2 shows that all the methods provide very good estimates of the frequencies and dampings. The results for moderate SNR in Table 3 show the improvement of Ibrahim's DLS algorithm over the LS algorithm. The LS algorithm is unable to identify the lowest frequency accurately. The IV results are better than either the LS or DLS estimates. The results for low SNR in Table 4 show the deterioration of both the LS and DLS estimates, while the IV results are reasonable, except for the first mode. All the numerical studies were done with 50 samples per data record. The effect of the noise bias can be seen clearly by viewing the LS estimates. The LS estimates of the damping factors are always too large. The DLS algorithm attempts to correct the bias by averaging the normal LS solution with the alternative LS solution, whose damping factor estimates are biased in the opposite direction.[3] The averaging results in good damping factor estimates for high to moderate SNR, but the bias of the alternative LS solution is too strong in the low SNR case and results in negative damping factor estimates.

Conclusions

The Ibrahim Time Domain Technique, using either the LS approach or the DLS approach, was shown to be biased, and the bias terms were presented. The preference for the DLS estimation over the LS estimation when there are moderate noise levels is demonstrated through numerical studies. Each approach considered in this study showed limited success in the case of low SNR. An Instrumental Variables approach was suggested to reduce the bias concerns. The improvement associated with the IV Method is evident for the moderate SNR case. There remains a practical concern with how to generate the instrumental variables, although any scheme where the instrumental variables are iteratively improved, similar to Ref. 4, may be realizable. Once the instrumental variables are generated, the computer cost associated with the IV method is approximately the same as that associated with the LS method (without extra computational modes), while the cost associated with the DLS method (without extra computational modes) is approximately double. Two iterations of an IV method could be performed for about the same computer cost as the DLS method (without extra computational DOF), and experience has shown that, in general, IV methods converge in only a few iterations. The cost of oversized LS and DLS methods increases exponentially as the number of computational modes increases. The relative computer cost between an iterative IV method and an oversized LS or DLS method depends upon the application. Reference 3 indicates the advantages of the DLS method over IV and Maximum Likelihood methods for applications with a large number of DOF.

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Reader's Forum

Brief discussion of previous investigations in the aerospace sciences and technical comments on papers published in the AIAA Journal *are presented in this special department. Entries must be restricted to a maximum of 1000 words, or the equivalent of one* Journal *page including formulas and figures. A discussion will be published as quickly as possible after receipt of the manuscript. Neither the AIAA nor its editors are responsible for the opinions expressed by the correspondents. Authors will be invited to reply promptly.*

Comment on "Solutions of One-Dimensional Steady Nozzle Flow Revisited"

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REFERENCE 1 discusses a number of topics dealing with one-dimensional nozzle flow. Two of these topics are also discusssed in Refs. 2 and 3 and may well be found in other compressible flow textbooks. These topics are a simple procedure for finding the nozzle solution when there is an internal shock wave, and the criteria for distinguishing different regimes for the flow in a converging/diverging nozzle.

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Reply by Author to G. Emanuel

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REGARDING two of many topics discussed in Ref. 1, I wish to thank G. Emanuel for bringing Refs. 2 and 3 to my attention. Indeed, an identical procedure for solving the exit Mach number with a normal shock in a nozzle appears not only in Refs. 1 (Remark 5) and 3 [p. 96, Eq. (7.7)] but also in Ref. 4 [p. 168, Eq. (5.12)]. As demonstrated in Ref. 1 (Remark 1), solution of the total pressure loss across the shock wave is equivalent and straightforward, yielding immediately the shock Mach number. As expressed in Eq. (14) of Ref. 1, the three critical points (called in Refs. 2 and 4) now can be described by the only two solutions of Eq. (13). Consequently, these two solutions completely delineate the seven regimes/points (e.g., see Refs. 2-4) associated with the convergent-divergent nozzle flows.

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Comment on "Influence of Initial and Boundary Conditions on Vortex Ring Development"

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IRDMUSA and Garris[1] found that vortex rings ejected from a circular hole (air in air) entrained less fluid than